5. Gaussian Beam Propagation

In this exercise, we simulate the propagation of Gaussian beams and the influence of their main parameters.

1. Install the GaussianBeam software.
    a. Initiate an *ideal* Gaussian beam with a wavelength of $\lambda = 600$ nm (orange), a beam diameter of $D = 250$ µm and a beam quality factor $M^2 = 1$.
    b. We are given lenses with focal lengths $f = 100$ mm, 50 mm, 20 mm, and 10 mm. Based on your prior knowledge, how should the waist diameter evolve as a function of these focal lengths?
    c. Check your assumptions using the software and conclude.
2. Change the initial beam diameter to $D = 500$ µm, what do you observe? Determine the waist diameters in this case for each of the lenses mentioned above.

Notes.

i. A **beam diameter** of $D = 250$ µm corresponds to a **beam waist** of $w_0 = 125$ µm ($D = 2 \cdot w_0$).

ii. Place both the "Input beam" and "Lens" at the same position (e.g. 0) to be sure the beam at the position where it encounters the lens has the diameter inputted in the table. Else, the beam will start to diverge and have a different diameter when it encounters the lens.

iii. The beam waist after the lens is indicated in the column "Waist (µm)" of the "Lens" line.

Based on your knowledge in optical engineering, you can guess that a more focusing lens (focal length smaller) will generate a more focused laser beam (waist smaller). The results for questions 1 and 2 are summarized here:

	$f = 100$ mm	$f = 50$ mm	$f = 20$ mm	$f = 10$ mm
$D = 250$ µm	$w = 96.7$ µm	$w = 65.2$ µm	$w = 29.7$ µm	$w = 15.2$ µm
$D = 500$ µm	$w = 73.1$ µm	$w = 37.8$ µm	$w = 15.3$ µm	$w = 7.6$ µm

Most importantly, keep in mind that the beam waist after a lens, w, is dependent on both the focal length f (increasing with it) and the input beam diameter D (decreasing against it).

3. How does the beam waist compare to the numerical aperture of a given objective?

The beam waist is actually inversely proportional to the Numerical Aperture (NA), equal to $n \cdot \sin(\theta)$ with n the refractive index of the propagation medium and θ the maximum acceptance half-angle.

$$NA = n \cdot \sin(\theta) = n \cdot \sin\left[\tan^{-1}\left(\frac{\frac{D}{2}}{f}\right)\right] \approx n\frac{D}{2f}$$

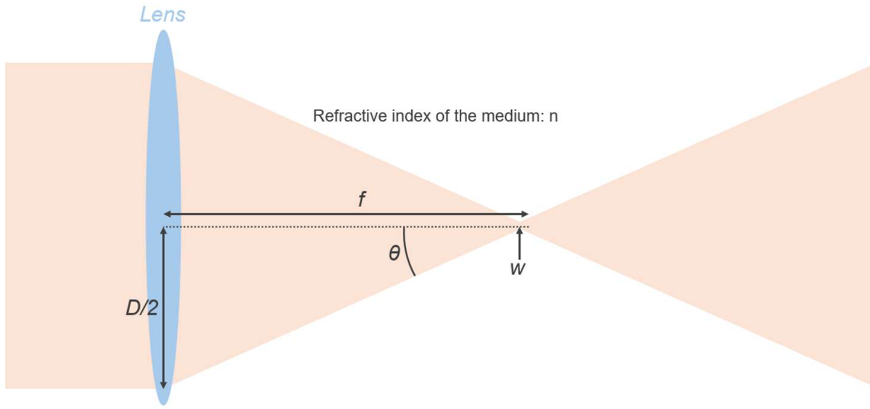


Illustration of the focusing of a beam through a lens.

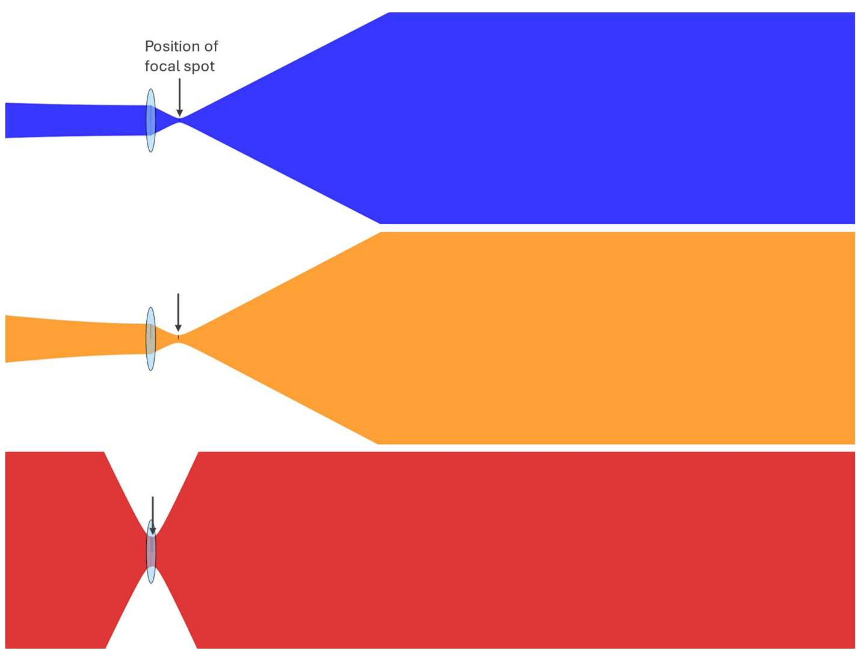
One can call the NA the "focusing power" of the optical system: the larger the NA, the better it can focus light (that is, the better the resolution). This behaviour is similar to the Abbe diffraction limit law, where the minimum resolvable distance between two features is limited by:

$$d = \frac{\lambda}{2 \cdot NA}$$

4. Now, set a beam diameter of $D = 250\ \mu\text{m}$ and a lens with a focal length $f = 20\ \text{mm}$. Observe the evolution of the beam propagation as a function of the following wavelengths: $600\ \text{nm}$, $300\ \text{nm}$ (UV-B), and $9999\ \text{nm}$ (LW-IR). What do you conclude?

$\boldsymbol{\lambda = 300\ \text{nm}}$	$\boldsymbol{\lambda = 600\ \text{nm}}$	$\boldsymbol{\lambda = 9999\ \text{nm}}$
$w = 15.2\ \mu\text{m}$	$w = 29.7\ \mu\text{m}$	$w = 121.4\ \mu\text{m}$

These quantities are proportional one to each other. The beam waist (and focal spot position) directly depends on the wavelength.



Beam waist and focal spot position for three different wavelengths:
300 nm *(blue), 600* nm *(orange) and 9999* nm *(red).*

5. Keeping the same conditions as for Question 4, with a wavelength of 600 nm, initiate a *real* Gaussian beam (that is, *non-ideal*). What do you observe?
   Hint. Which parameter should you change to describe a realistic Gaussian beam?

The output from real-life lasers is not truly Gaussian (although the output of a single mode fibre is a very close approximation). To accommodate this variance, the quality factor has been defined to describe the deviation of the laser beam from a theoretical Gaussian. For an ideal Gaussian, we have seen that $M^2 = 1$. For a real laser beam, $M^2 > 1$.

The quality factor of HeNe (helium neon) lasers is typically $M^2 < 1.1$. For high energy multimode lasers, the M^2 can be as high as 25 or 30. In all cases, the quality factor affects the characteristics of a laser beam and cannot be neglected in optical designs.

Let us compare the ideal situation with a real-world situation where $M^2 = 1.5$. We keep a beam diameter of $250\ \mu m$ and 600 nm wavelength, and a lens with $f = 20$ mm.

$\boldsymbol{M^2 = 1.0}$	$\boldsymbol{M^2 = 1.5}$
$w = 29.7\ \mu m$	$w = 43.0\ \mu m$

The beam waist is roughly proportional to the quality factor, which is a dimensionless value representing the beam quality. Again, a perfectly Gaussian beam (single mode TEM_{00}) has, by definition, $M^2 = 1$.

To summarize, M^2 factor determines the beam's quality in terms of focusability, or how well a divergent laser source can be collimated. The ISO Standard 11146 defines the M^2 factor as:

$$M^2 = \frac{\pi w_0 \theta}{\lambda}$$

with w_0 the beam waist, θ the divergence angle of the laser, and λ the lasing wavelength.

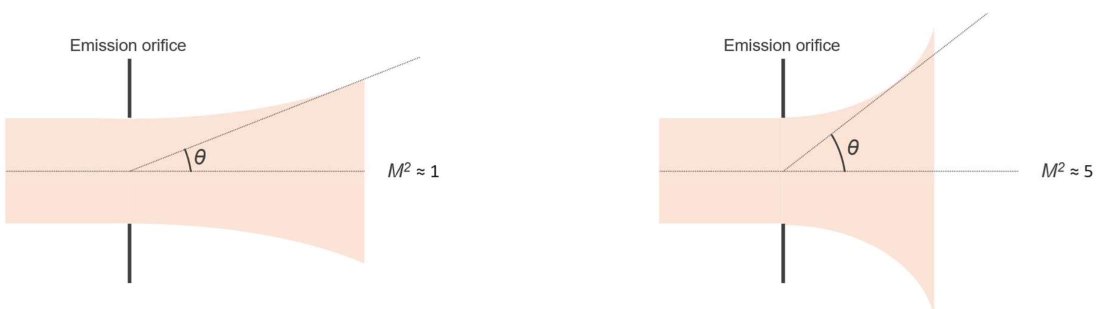


Illustration of the effect of M2 factor on beam quality for a close to perfect beam (left) and a beam with high M2 factor (right).

6. Material Removal Rate in Laser Processing

We consider a pulsed laser. The pulses are emitted at a frequency f. This frequency is commonly called *repetition rate*. Furthermore, we assume that the laser at focus has a spot radius of w_0, and each single pulse carries an energy E_p. Finally, we assume that the ablation depth *per pulse* is z_a.

1. How would you define the *material removal rate* in the case of an ablation process? Give an expression for this MRR.
   Note. In the jargon of laser processing, the material removal rate is referred to as ablation rate.

The material removal rate is usually defined as "how much volume of material is removed by unit of time". Its unit is typically $\mathrm{mm}^3/\mathrm{min}$ or mm^3/s.
Knowing the ablation depth per pulse z_a, we can deduce the volume ablated by one single pulse:

$$MRR_{pp} = \pi w_0^2 z_a$$

Knowing the pulse repetition rate f, we can write the MRR as following:

$$MRR = MRR_{pp} f = \pi w_0^2 z_a f$$

2. How would you define the *MRR energy efficiency* (i.e., "how much volume of material is removed per spent unit of energy")? Give an expression for this MRR_E.

We can define the MRR_E as "the material removal rate divided by the used optical power $P = E_p f$", its unit being mm^3/J:

$$MRR_E = \frac{MRR}{E_p f} = \frac{\pi w_0^2 z_a}{E_p}$$

Application. Let us consider the case of cutting a polymer (ABS) with a UV laser. We provide the following parameters: $MRR_E = 0.067\ \mathrm{mm}^3/\mathrm{J}$, $f = 30\ \mathrm{kHz}$, scanning speed $v = 10\ \mathrm{mm/s}$, average optical power $P = 20\ \mathrm{mW}$, and $2w_0 = 11\ \mu\mathrm{m}$, the spot diameter. We assume that the spot diameter stays constant over the thickness of the considered material.

3. Calculate the ablation rate per pulse and the ablation rate.

Using the average power, we can calculate the pulse energy: $E_p = P/f \cong 0.67\ \mu\mathrm{J}$.
Then, using the MRR_E, we can get the volume ablated by one single pulse:

$$MRR_{pp} = MRR_E \cdot E_p \cong 4.5 \cdot 10^{-8}\ \mathrm{mm}^3$$

The ablation rate is hence $MRR = MRR_{pp} \cdot f \cong 1.4 \cdot 10^{-3}\ \mathrm{mm}^3/\mathrm{s}$.

4. How long would it take to cut a $c = 1\ \mathrm{mm}$ square out of a $t = 200\ \mu\mathrm{m}$ thick substrate?

Knowing the ablation rate, we need to estimate the time required to ablate the volume corresponding to the contour of the square over the substrate's thickness, with a cutting thickness of $11\ \mu\mathrm{m}$ (the waist).

As a first assumption, we neglect the pulse-to-pulse effects of saturation and incubation. This is an oversimplification as these effects tend to become strong at high repetition rates (and/or slow scanning speed). Thermal accumulation might also occur, which typically disrupts the ablation process.

Under those simple assumptions, we can calculate the volume that needs to be ablated:

$$V = 4 \cdot c \cdot t \cdot 2w_0 = 4 \cdot 1 \cdot 0.200 \cdot 0.011 = 0.0088\ \mathrm{mm}^3$$

The time needed to remove this volume is therefore:

$$t = \frac{V}{MRR} \cong 6.3\ \mathrm{s}$$

With the scanning speed v, we deduce that the laser path length is about

$$l_{path} = t \cdot v = 65\ \mathrm{mm}$$

Note that this is about 16 times the contour of the square, which means that the z dimension of the laser spot is around $13\ \mu\mathrm{m}$… These results are valid assuming that a z-axis platform is adjusting the height of the beam waist so that the beam diameter stays constant during the process.

7. Glass Manufacturing with Femtosecond Lasers

(by Antoine Duret)

7.1. Femtosecond Laser Printing

1. List the benefits and drawbacks of such a process in comparison to other laser-based manufacturing technologies.

This process has several unique capabilities:

- It relies on direct laser writing and hence does not require masks nor clean rooms.
- It is a two-step process, avoiding any complex assemblies, bonding, and post-processing steps.
- It allows quick device optimizations for faster market entry and rapid prototyping.
- It is versatile and can be applied to a broad range of microtechnologies, including precision mechanisms, optical communications, microfluidic chips, biomedical devices, and packaging.
- Complex 3D geometries can be printed, with multiple slopes on the same part, cavities, channels, excellent resolution ($1\ \mu\text{m}$) and very high aspect ratio ($> 1:500$).

The main drawback is the processing time, which increases with the complexity of the part. It is the sum of the laser writing time and etching time (but several devices can be etched together in parallel).

Another limitation is the choice of materials, restricted to glasses (and even not all of them).

2. Glass appears as an ideal material because of its transparency at $1030\ \text{nm}$, the wavelength of the femtosecond laser. List other reasons making it very relevant for microengineering applications.

Glass is isotropic, chemically stable, and biocompatible, it is a good electrical insulator, it offers excellent mechanical, optical and thermal properties and it has a high durability.

3. What glass property may act as a troublemaker if the process is not well controlled?

Glass is a brittle material. Even if it is flexible if sufficiently thin, it can break very easily as it does not sustain any plastic deformation. This can cause many issues, including chipping during the etching due to residual constraints caused by the laser writing step.

4. The femtosecond laser pulses induce a change of volume in the matter. What are the three consequences resulting from this volume change? Why are they important?
Hint. Consider one mechanical consequence, one optical consequence and one process-related consequence.

The change of volume increases the stress, alters the refractive index, and increases the etching rate. Stress increase must be considered to avoid chipping or part failure during or after manufacturing. The change of refractive index must be considered when ordering the laser operations, so that the laser beam never passes again through a written area (to avoid light refraction / diffusion and subsequent manufacturing issues). Finally, the etching rate increase in the written areas is the key to be able to obtain a part with the expected geometry after the etching. All the surfaces would etch at the same speed otherwise!

7.2. Case Study of a Glass Package

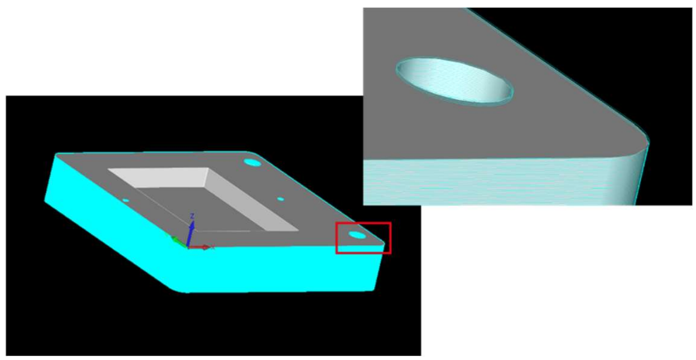
5. Knowing that the laser beam comes from the bottom, explain how you would proceed to write the geometry in a glass slide of desired thickness.

The only rule is simply to avoid rewriting through an already written area.

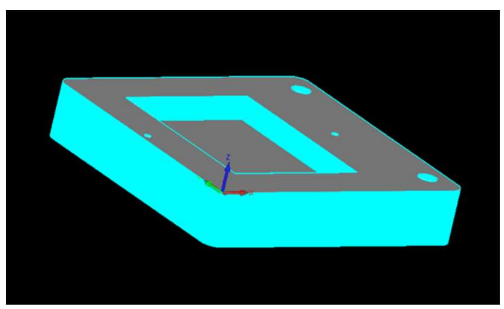
(1) We can start by writing the closed contours of the part, always from top to bottom. This includes the external border of the part, the two traversing holes and the two small alignment holes.

(2) We can then write the inner closed contour defining the pocket, from top to bottom, until the desired depth.

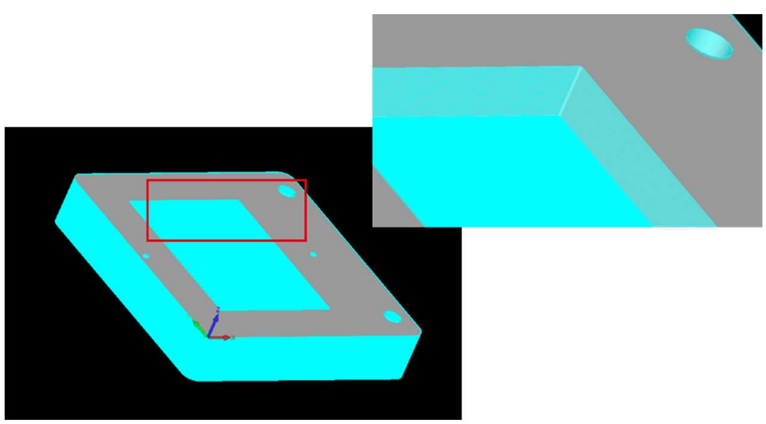
(3) Finally, we need to write the full pocket plane, with a single line following a back-and-forth pattern. Remember that you have to write all the surfaces where you want to remove matter.



Step (1) described in Question 5.



Step (2) described in Question 5.



Step (3) described in Question 5.

6. Knowing that the vertical pitch is $p_v = 15\ \mu\text{m}$, how many laser passes do you have along the full height of the part?

$$N_{passes} = \frac{t}{p_v} = \frac{5.8}{0.015} \cong 386$$

7. Femtosecond lasers have very specific parameters. In our case, knowing that the laser has a repetition rate $RR = 1000\ \text{kHz}$, a peak power $P_{peak} = 1.53\ \text{MW}$ and a pulse duration $PD = 150\ \text{fs}$, compute the average power P_{av} of the light source.

The pulse energy is first given by

$$PE = P_{peak} \cdot PD = 1.53 \cdot 10^6 \cdot 150 \cdot 10^{-1} \quad \cong 230\ \text{nJ}$$

From this result, we can compute the average power:

$$P_{av} = RR \cdot PE = 1000 \cdot 10^3 \cdot 230 \cdot 10^{-9} \cong 0.23\ \text{W}$$

We see that despite extremely high peak powers, the average power over time is very small. This is why there is no heat transfer in the matter in the non-cumulative regime, hence no change of state and very clean writing. This also explains why femtosecond lasers are used to perform surgeries directly into the human eye... Finally, we might be surprised by the very low electrical energy cost required to supply a femtosecond laser used in such conditions!

7.3. Selectivity in Etching

8. What is the minimum etching time $t_{etc\ \ ,min}$ required for our package?

First, identify and compute the critical etched length l_c in the package. In our case, the critical length is the shortest distance that the acid solution has to cover to reach the middle of the pocket plane (that is, the longest distance to cover over the whole part). Assuming that the pocket is written with lines parallel to its short side:

$$l_c = 2.5 + \frac{14.8}{2} = 9.9\ \text{mm}$$

The minimum etching time is then given by:

$$t_{etch,min} = \frac{l_c}{w} = \frac{9.9}{0.350} \cong 28.3\ \text{h}$$

9. What is the maximum parasitic etching d_p in that case?

All the unwritten surfaces will also be etched of a quantity given by

$$d_p = t_{etch,min} \cdot i = 28.3 \cdot 0.5 \cong 14.2\ \mu\text{m}$$

10. Detail two strategies to minimize this parasitic effect and reach dimensions closer to the expected ones.

Strategy 1. We can add *hatching lines* over the pocket. They will increase the laser writing time but reduce the l_c and hence the required etching time (and undesired etching). To help the etching solution reach the remote areas of the pocket, the hatching lines must be orthogonal to the lines defining the pocket plan itself.

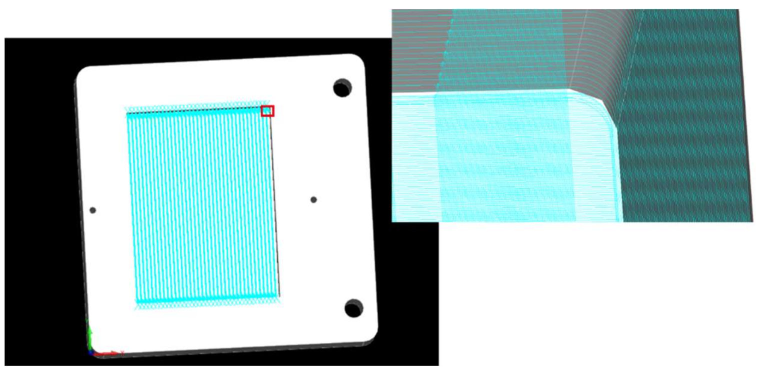
Let us fix a hatching distance equal to $400\ \mu m$. The new critical length, minimum etching time and parasitic etched distance are:

$$l_c = 2.5 + \frac{0.400}{2} = 2.7 \text{ mm}$$

$$t_{etch,min} = \frac{2.7}{0.350} \cong 7.7 \text{ h}$$

$$d_p = 7.7 \cdot 0.5 \cong 3.9 \text{ µm}$$

This technique will also facilitate the detaching of the bulk material to remove to open the pocket.



The hatching operation. Chronologically, this step shall be added between the writing of the edges of the pocket and the writing of the pocket plane.

Strategy 2. In complement, we can take into account the unwanted etching in the toolpath design itself and write all the geometries with dimensions extended by this d_p. The part is written bigger than its correct dimensions to balance the parasitic etching.

11. Evaluate the verticality of the traversing holes.

We know that the part has a thickness $t = 5.8$ mm. The etching solution comes from both sides of the holes, so the critical etched distance referred to such hole is $l_{c,hole} = 2.9$ mm. In other words, the time interval between the etching of the external and middle heights of the holes is:

$$t_{etch,hole} = \frac{l_{c,hole}}{w} = \frac{2.9}{0.350} \cong 8.3 \text{ h}$$

During 8.3 h, the extremities of the hole are etched with an etch rate i while the rest is etched with the written etch rate w. At the end, the extremities of the holes are over-etched by a distance:

$$d_{p,hole} = t_{etch,hole} \cdot i = 8.3 \cdot 0.5 \cong 4.2 \text{ µm}$$

The slope of the inner surface of the holes is thus:

$$\alpha = atan\left(\frac{d_{p,hole}}{l_{c,hole}}\right) = atan\left(\frac{4.2}{2900}\right) \cong 0.083°$$

12. Empirical data show that the etching time typically improves the failure strength, allowing some parts to withstand stresses more than 2 GPa above the theoretical yield strength of glass. Can you explain why?

This is another interesting effect occurring during etching: the removal of surface defects and broken bonds. These elements create regions with stress concentration which are conducive to cracks nucleation and chipping. This is why improvements in the mechanical properties of etched glass parts were observed as a function of the etching time.

7.4. Welding the Package

13. *(Advanced)* Think about what could go wrong with the package studied in this exercise, knowing that we would like to flip another identical part on top of the first one and laser-weld them together to form a full package, as shown in **Figure 9**. Think about the stress concentrations and their implications. Once you have identified at least two potential issues, propose some ideas to solve them.

Because of the substantial volume changes induced in the matter by the pocket writing (it has very large dimensions!), the full part may slightly bend during the laser step and may not detach correctly at the end of the etching step. A solution to this problem would be to increase the etching time as long as necessary. This can always be done, provided that the tolerances are not critical or that the design is compensated accordingly.

However, to be able to perform the welding properly, the surfaces to weld need to be perfectly flat to guarantee excellent contact between them. Several options are possible to minimize the bending of the part due to internal constraints: decrease the size of the pocket (if possible), reduce the pulse energy (but the writing step might become extremely long), or decrease the pitch at the bottom of the part to kind of counterbalance the internal constraints.

Reducing the internal constraints is also important to avoid any chipping along the pocket borders. Sharp edges and angles are perfect spots for stress concentration. Chamfering or filleting these regions might be a good solution.

Finally, a good practice when manufacturing glass parts with lasers is always to start the operations a few tens of microns above the actual surface of the part and finish a few tens of microns below the actual bottom of the part. This allows us to be sure that the part will be written over the full height or thickness of the substrate, despite eventual intrinsic or appearing nonplanarities.

8. Cutting Strategy in Blanking

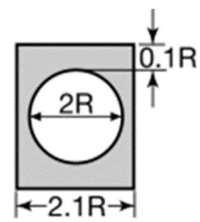
1. *Qualitative Assessment.* Intuitively, which of the two strategies will yield the lowest amount of scrap metal? What are the limiting factors determining the minimum distance between holes?

Intuitively, it is not difficult to guess that the second strategy with multiple rows of circles cut out of a single strip will be more material-saving. The grey surface corresponding to scrap material can be minimized compared to the first strategy.

The distance between holes cannot be too small, or the blanking process will not be performed properly anymore. If the cutting force exceeds the shear force that the beam of length the minimum distance between two holes can sustain, the strip will fail. Therefore, this distance varies as a function of the material being cut (in terms of its mechanical properties, but also of its thickness).

2. *Quantitative Assessment.* For both cases, estimate the percent scrap in producing round blanks if the clearance between blanks is one tenth of the radius of the blank (i.e., 10% of the disk radius needs to be preserved during cutting). Conclude.
Hint. Draw a repeating unit cell.

Single row solution. A repeating unit cell for the single row blanking is illustrated here. The area of the unit cell is

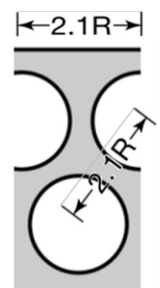


$$A_{uc} = (2.2R)(2.1R) = 4.62R^2$$

The area of the circle is $A_{hole} = 3.14R^2$.
Therefore, the scrap is:

$$scrap = \frac{4.62R^2 - 3.14R^2}{4.62R^2} \cdot 100 = 32\%$$

Multiple row solution. A repeating unit cell for the multiple row blanking is illustrated here. We know that the distance between the edge of a hole and the metal strip border is $0.1R$. We also know that the distance between the centers of two holes is $2.1R$, meaning that the width of the cell is $w = 2.1R$. We can find the height of the cell simply using Pythagoras' theorem:



$$h = 0.1R + R + \left(\sqrt{w^2 - \left(\frac{w}{2}\right)^2}\right) + R + 0.1R = 2.2R + h'$$

with $h' = w/2$ being the distance between the first and second row of holes (at the level of their centers).

Using the same approach than before, it can be shown that scrap material in the case of the multiple row blanking is about 26%.

3. Calculate the blanking force for punching a disk of 5 mm in diameter out of a 0.5 mm-thick strip of spring steel ($s_L = 400$ MPa, t_L » $0.5s_L$).

The blanking force is $F_b = perimeter \times plate\ thickness \times shear\ strength$:

$$F_b = (\pi \cdot 5 \cdot 10^{-3}) \cdot 0.5 \cdot 10^{-3} \cdot (0.5 \cdot 400 \cdot 10^6) = 1570 \text{ N}$$

9. Folding

9.1. Preliminary Work

1. Demonstrate the formula shown in class that expresses the strain in a beam curved in pure bending. Assume that the curvature remains big compared to the beam thickness, so that the neutral line is passing exactly through the middle of the beam.
   Hint. Make a proper drawing and use the basic definition of the engineering strain.

The engineering strain is defined as:

$$\varepsilon = \frac{\Delta L}{L_0}$$

Let us apply here to the arc length s as shown in the figure. This arc is defined by the angle q. In this loading case, the strain remains zero along the neutral axis. Let us now call s_0, the arc length defined on the neutral line for the angle q as well. By definition, we have:

$$s_0 = \rho\theta$$

The radius of curvature at a given position z in the beam is given by:

$$\rho(z) = \rho + z$$

z being the distance from the neutral plane (see the figure).

Therefore, the length of the arc corresponding to the angle q at a position z within the beam writes:

$$s(z) = \rho(z)\theta = (\rho + z)\theta$$

The arc length $s(z)$ can be seen as the 'deformed' arc s_0 at position z. Therefore, using the strain definition:

$$\varepsilon(z) = \frac{s(z) - s_0}{s_0} = \frac{z}{\rho}$$

with $0 \leq z \leq \frac{t}{2}$.

The strain is maximum at the inner and outer surface of the beam.

9.2. Plastic Deformation[7]

2. We bend the beam. Knowing the plastic stress limit σ_y for a given material, calculate the radius of curvature – we will call it R_p – for which plasticity in the beam will start to appear.

Plasticity occurs when the stress in the beam exceeds the elastic limit of the material. Since the maximal stress when bending is present at the outer surfaces, the condition could be expressed as:

$$\sigma\left(z = \frac{t}{2}\right) = E\frac{t}{2\rho} > \sigma_y$$

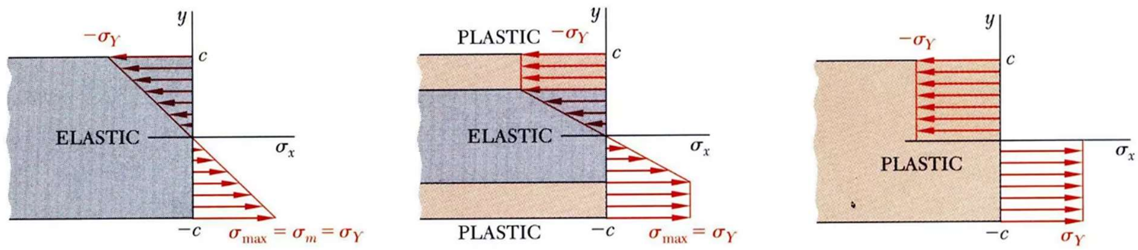
In other words, this condition is found when the bending radius inducing plastic deformation is:

$$R_p = E\frac{t}{2\sigma_y}$$

[7] *Adapted from Beer, Johnston, DeWolf,* Mechanics of materials, *3rd Ed., The McGraw-Hill Companies.*

3. Describe how the stress state evolves in the material as the beam is bent beyond R_p. What happens during unloading? Use drawings to illustrate the evolution of the stress state and assume that the deformation remains elastoplastic.

As the beam is bent beyond R_p, the material passes from elastic to elastoplastic deformation. Plastic zones develop around an elastic core. If the moment is increased further, the elastic core goes to zero, corresponding to a fully plastic deformation.

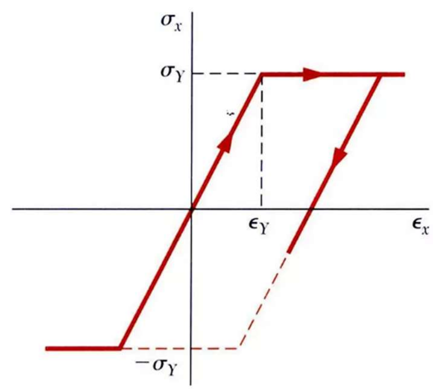


Plastic zones develop in a part made of an elastoplastic material if the bending moment is large enough.

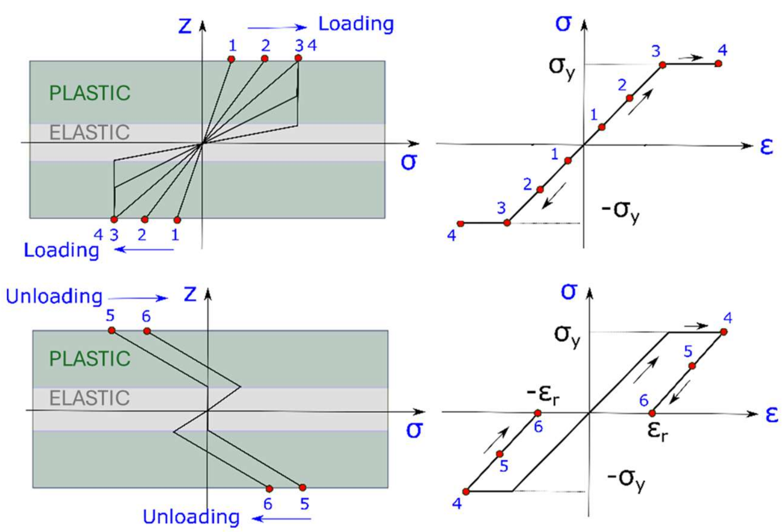
Since the linear relation between normal stress and strain applies at all points during the *unloading* phase, unloading may be handled by assuming the member to be fully elastic. Tracing a stress-strain graph, we see that residual stresses appear.

They are obtained by applying the principle of superposition to combine the stresses due to loading with a moment M (elastoplastic deformation) and unloading with a moment $-M$ (elastic deformation).

In general, in case of plastic deformation, the final value of stress at a point will not be zero.



Let us summarize:



(1), (2) Loading: elastic deformation until the yield strength

(3), (4) Saturation of the stress at the yield strength: elastoplastic deformation

(5), (6) Unloading: elastic return with residual stresses (the plastic region is permanently deformed with respect to the elastic core, inducing stresses).

Note the sign of the curves which is inversed since the applied moment is also inversed during unloading.

***Note.** The stress states through the cross-section are represented when the loading has reached step (4) on the graphs!*

Application. A uniform rectangular beam is subjected to a bending moment $M = 36.8\ \mathrm{kN \cdot m}$. It is made of an elastoplastic material with $\sigma_y = 240\ \mathrm{MPa}$ and $E = 200\ \mathrm{GPa}$.

4. Determine the thickness of the elastic core t_Y and the radius of curvature of the neutral surface, knowing that the applied moment and maximum moment for elastic bending are linked as:

$$M = \frac{3}{2} M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2}\right)$$

Using the provided equation and rearranging the terms, we obtain:

$$y_Y = \sqrt{3c^2 \left(1 - \frac{2}{3} \frac{M}{M_Y}\right)}$$

The maximum elastic moment is given by:

$$M_Y = I \frac{\sigma_Y}{c} = \frac{b(2c)^3}{12} \frac{\sigma_Y}{c} = 28.8 \text{ kN}$$

So the half thickness of the elastic core is

$$y_Y = \sqrt{3 \cdot 60^2 \cdot \left(1 - \frac{2}{3} \cdot \frac{36.8}{28.8}\right)} = 40 \text{ mm}$$

and $t_Y = 2y_Y = 80 \text{ mm}$. The radius of curvature is given by:

$$\varepsilon_Y = \frac{\sigma_Y}{E} = 1.2 \cdot 10^{-3}, \qquad \varepsilon_Y = \frac{y_Y}{\rho} \Rightarrow \rho = \frac{40 \cdot 10^{-3}}{1.2 \cdot 10^{-3}} = 33.3 \text{ m}$$

5. After unloading the beam (loading reduced back to zero), determine the distribution of residual stresses (draw the correspond graphs) and the radius of curvature of the neutral surface (i.e., at the edge of the elastic core).

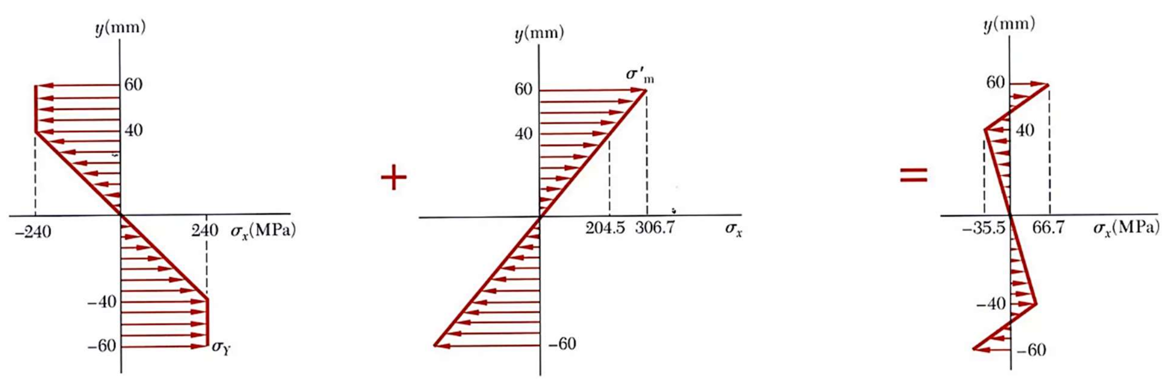
Let us consider each step one by one (remember what we discussed in Question 3):

- *Loading*. For $M = 36.8 \text{ kN} \cdot \text{m}$, $y_Y = 40 \text{ mm}$ and $\sigma_x(y_Y) = \sigma_y = 240 \text{ MPa}$
- *Unloading*. For $M = -36.8 \text{ kN} \cdot \text{m}$, $\sigma'_m = \frac{Mc}{I} = 306.7 \text{ MPa} < 2\sigma_Y$ and $\sigma'_x(y_Y) = 204.5 \text{ MPa}$
- *Final state: principle of superposition*. Finally, for $M = 0$, we have

$$\varepsilon_{x,el.} = \frac{\sigma_{x,min}}{E} = \frac{\sigma'_x(y_Y) - \sigma_x(y_Y)}{E} = \frac{-35.5 \cdot 10^6}{200 \cdot 10^9} = -177.5 \cdot 10^{-6}$$

$$\varepsilon_{x,pl.} = \frac{\sigma_{x,max}}{E} = \frac{\sigma'_m - \sigma_x(y_Y)}{E} = \frac{66.7 \cdot 10^6}{200 \cdot 10^9} = 333.5 \cdot 10^{-6}$$

The graphs representing the situation can be drawn as follows:



We thus obtain **at the edge of the elastic core:**

$$\rho = -\frac{y_Y}{\varepsilon_{x,el.}} = \frac{40 \cdot 10^{-3}}{177.5 \cdot 10^{-6}} = 225 \text{ m}$$

9.3. Hinge Forming

6. Calculate the initial bend radius R_i.

As the die has a diameter of $\emptyset_{die} = 20.0$ mm and the sheet thickness is $t = 1.00$ mm, the initial bend radius is:

$$R_i = \frac{20.0}{2} - 1.00 = 9.00 \text{ mm}$$

7. Considering the spring back effect, what will be the outside diameter $\emptyset_f$ of the hinge once released from the die?
Hint. The following formula expresses the final bent radius as a function of the initial bend radius:

$$\frac{R_i}{R_f} = 4\left[\frac{R_i \sigma_y}{Et}\right]^3 - 3\left[\frac{R_i \sigma_y}{Et}\right] + 1$$

Let us start by evaluating the common term:

$$\frac{R_i \sigma_y}{Et} = \frac{9.00 \cdot 90 \cdot 10^6}{1.00 \cdot 70 \cdot 10^9} \cong 0.0116$$

We thus obtain for the equation:

$$\frac{R_i}{R_f} = 4 \cdot 0.0116^3 - 3 \cdot 0.0116 + 1 \cong 0.965 \Rightarrow R_f = \frac{9.00}{0.965} \cong 9.33 \text{ mm}$$

the final bend radius. From this result, we compute the final outside diameter:

$$\emptyset_f = 2(R_f + t) = 20.66 \text{ mm}$$

8. What is the amount of permanent strain in the final shape? Assume that the bending radius is still enough large compared to the beam thickness so that the neutral line is still in the middle.

In the first part of this exercise, we have seen that $\varepsilon(z) = z/\rho$. The **maximum** amount of permanent strain can hence be estimated by taking $\rho = \emptyset_f/2$:

$$\varepsilon_p = \frac{t}{\frac{\emptyset_f}{2}} = \frac{2 \cdot 1.00}{20.66} \cong 0.0968$$

so about 9.68% of maximum plastic deformation.